Quantum Number Fractionalization in Antiferromagnets *

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Contents

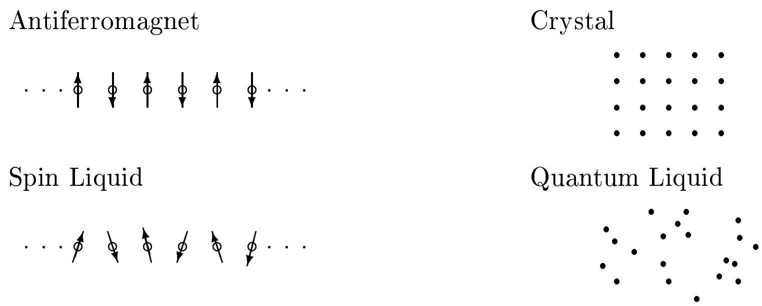

Figure 1: Illustration of the quantum spin liquid ground state.

1 Introduction

In these lectures we shall derive and discuss a set of exact eigenstates of the Haldane-Shastry [1, 2] model, a realization of the spin-1/2 Heisenberg chain in which the quantum-disordered spin liquid ground state and the neutral, spin-1/2 excitations of such systems are particularly easy to understand. This behavior is not unique to the model, and in particular occurs in the Bethe solution of the near-neighbor Heisenberg chain, where it was first discovered [3, 4], but it is more accessible in this form. The model also makes the relationship of the spin chain to the fractional quantum Hall effect transparent [5]. The key results are these:

1. The ground state has the same functional form as the fractional quantum Hall ground state.
2. This ground state has quantum disorder and has the same relation to the antiferromagnetically ordered state that a quantum liquid has to a conventional crystal.
3. The elementary excitations of this state are spin-1/2 particles, spinons, and not spin waves, which are the elementary excitations of an ordered antiferromagnet. Spinons have
   (a) A relativistic band structure with a Dirac point at momentum $\pm\pi/2$.
   (b) 1/2 fractional statistics.
   (c) $N/2$ allowed momenta rather than N.
4. The ground state is effectively 2-fold degenerate. The missing N/2 states of the spinon are excitations of the other ground state.
5. The Hamiltonian has a factorization which may be construed as supersymmetric.

2 Haldane-Shastry Hamiltonian

Let a lattice of N sites be wrapped onto a unit circle, as shown in Fig. 2, so that each site may be expressed as a complex number z_α satisfying

$$z_\alpha^N - 1 = 0 \quad , \tag{1}$$

and let each site possess a single unpaired half-integral spin with its corresponding spin operator $\vec{S}_\alpha$. The Haldane-Shastry Hamiltonian is then given by

$$\mathcal{H}_{HS} = J \left(\frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta}^{N} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} \quad . \tag{2}$$

This Hamiltonian is translationally invariant, satisfies

$$[\mathcal{H}_{HS}, \vec{S}] = 0 \qquad \vec{S} = \sum_{\alpha}^{N} \vec{S}_\alpha \quad , \tag{3}$$

and possesses the special internal symmetry

$$[\mathcal{H}_{HS}, \vec{\Lambda}] = 0 \qquad \vec{\Lambda} = \frac{i}{2} \sum_{\alpha \neq \beta} \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) (\vec{S}_\alpha \times \vec{S}_\beta) \quad . \tag{4}$$

Proof

We proceed by applying the commutation relations

$$[\vec{S}_1, (\vec{S}_1 \cdot \vec{S}_2)] = i(\vec{S}_1 \times \vec{S}_2) \tag{5}$$

$$[(\vec{S}_1 \times \vec{S}_2), (\vec{S}_1 \cdot \vec{S}_2)] = \frac{i}{2}(\vec{S}_1 - \vec{S}_2) \quad , \tag{6}$$

the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} \quad , \tag{7}$$

the fact that

$$\left[\frac{z_j + z_k}{z_j - z_k} - \frac{z_\ell + z_k}{z_\ell - z_k} \right] \frac{1}{|\, z_j - z_\ell\,|^2} = \frac{z_j z_k z_\ell}{(z_j - z_k)(z_\ell - z_k)(z_j - z_\ell)} \quad , \tag{8}$$

which is totally antisymmetric under permutation of j, k, and ℓ, and

$$\sum_{j \neq k} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|\, z_j - z_k\,|^2} = - \sum_{\alpha=1}^{N-1} \frac{z_\alpha (z_\alpha + 1)}{(z_\alpha - 1)^3} = 0 \quad . \tag{9}$$

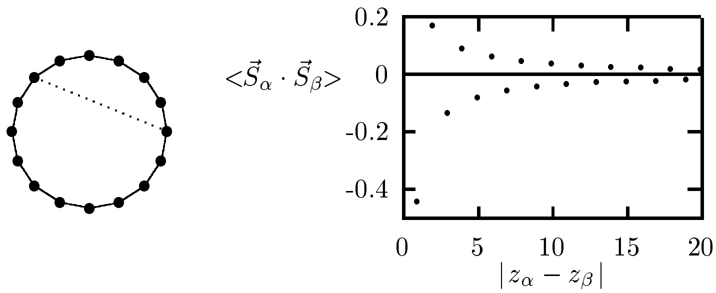

Figure 2: Left: Illustration of Haldane-Shastry model. Right: Magnetic correlation function defined by Eq. (28).

We have

$$\sum_{j\neq k}\sum_{\alpha\neq\beta} \frac{z_j+z_k}{z_j-z_k}\frac{1}{|z_\alpha-z_\beta|^2}\;[(\vec{S}_j\times\vec{S}_k),(\vec{S}_\alpha\cdot\vec{S}_\beta)]$$

$$= 4i\sum_{j\neq k\neq \ell} \frac{z_j+z_k}{z_j-z_k}\frac{1}{|z_j-z_\ell|^2}\left[(\vec{S}_j\cdot\vec{S}_k)\;\vec{S}_\ell-(\vec{S}_\ell\cdot\vec{S}_k)\;\vec{S}_j\right]$$

$$+\, i\sum_{j\neq k} \frac{z_j+z_k}{z_j-z_k}\frac{1}{|z_j-z_k|^2}\;(\vec{S}_j-\vec{S}_k)=0 \quad . \;\square \tag{10}$$

It follows from the commutation relations

$$[S^a,S^b]=i\,\epsilon^{abc}S^c \qquad\qquad [S^a,\Lambda^b]=i\,\epsilon^{abc}\Lambda^c \quad , \tag{11}$$

that $\mathcal{H}_{HS}$, S^2, and $(\vec{\Lambda}\cdot\vec{S})$ all commute with each other.

3 Ground State

Let us imagine the spin system to be a 1-dimensional string of boxes populated by hard-core bosons, the $\downarrow$ spin state corresponding to an empty box and the $\uparrow$ spin state corresponding to an occupied one. The ground state wavefunction is a rule by which a complex number is assigned to each boson configuration. The total number of bosons is conserved, as it is physically the same thing as the eigenvalue of S^z. Let N be even and let $z_1, ..., z_{N/2}$ denote the locations of $N/2$ bosons, the number appropriate for a spin singlet. Then the Haldane-Shastry ground state is

$$\Psi(z_1, ..., z_{N/2}) = \prod_{j<k}^{N/2} (z_j - z_k)^2 \prod_{j=1}^{N/2} z_j \quad . \tag{12}$$

Its energy is

$$\mathcal{H}_{HS} |\Psi> = -J \left(\frac{\pi^2}{24} \right) \left(N + \frac{5}{N} \right) |\Psi> \quad . \tag{13}$$

Proof

We begin by observing that $[S_\alpha^+ S_\beta^- \Psi](z_1, ..., z_{N/2})$ is identically zero unless one of the arguments $z_1, ..., z_{N/2}$ equals z_α. We have

$$[\left\{ \sum_{\beta \neq \alpha}^{N} \frac{S_\alpha^+ S_\beta^-}{|z_\alpha - z_\beta|^2} \right\} \Psi](z_1, \ldots, z_{N/2})$$

$$= \sum_{j=1}^{N/2} \sum_{\beta \neq j}^{N} \frac{1}{|z_j - z_\beta|^2} \Psi(z_1, \ldots, z_{j-1}, z_\beta, z_{j+1}, \ldots, z_{N/2})$$

$$= \sum_{j=1}^{N/2} \sum_{\ell=0}^{N-2} \left\{ \sum_{\beta \neq j}^{N} \frac{z_\beta (z_\beta - z_j)^\ell}{\ell! \, |z_j - z_\beta|^2} \right\} (\frac{\partial}{\partial z_j})^\ell \left\{ \Psi(z_1, \ldots, z_{N/2}) / z_j \right\}$$

$$= \sum_{j=1}^{N/2} \left\{ \frac{(N-1)(N-5)}{12} z_j - \frac{N-3}{2} z_j^2 \frac{\partial}{\partial z_j} \right.$$

$$\left. + \frac{1}{2} z_j^3 \frac{\partial^2}{\partial z_j^2} \right\} \left\{ \Psi(z_1, \ldots, z_{N/2}) / z_j \right\}$$

$$= \left\{ \frac{N(N-1)(N-5)}{24} - \frac{N-3}{2} \sum_{j \neq k}^{N/2} \frac{2 z_j}{z_j - z_k} \right.$$

$$\left. + \frac{1}{2} \left[\sum_{j \neq k \neq m}^{N/2} \frac{4 z_j^2}{(z_j - z_k)(z_j - z_m)} + \sum_{j \neq k}^{N/2} \frac{2 z_j^2}{(z_j - z_k)^2} \right] \right\} \Psi(z_1, \ldots, z_{N/2})$$

$$= \left\{ \frac{N(N-1)(N-5)}{24} - \frac{N(N-3)}{4}(\frac{N}{2}-1) + \frac{1}{2}\left[\frac{2N}{3}(\frac{N}{2}-1)(\frac{N}{2}-2) \right.\right.$$

$$\left.\left. + \frac{N}{2}(\frac{N}{2}-1) - \sum_{j \neq k}^{N/2} \frac{2}{|\, z_j - z_k \,|^2} \right] \right\} \Psi(z_1, \ldots, z_{N/2})$$

$$= \left\{ -\frac{N}{8} - \sum_{j \neq k}^{N/2} \frac{1}{|\, z_j - z_k \,|^2} \right\} \Psi(z_1, \ldots, z_{N/2}) \quad . \tag{14}$$

Here we have used the fact that

$$\frac{1}{|\, z_\alpha - z_\beta \,|^2} = -\frac{z_\alpha z_\beta}{(z_\alpha - z_\beta)^2} \quad , \tag{15}$$

i.e. that the Hamiltonian is effectively analytic in the spin coordinates, and the sums (with sites labeled so that $z_N = 1$)

$$\sum_{\alpha=1}^{N-1} \frac{1}{|\, z_\alpha - 1 \,|^2} = \frac{N^2-1}{12} \quad , \tag{16}$$

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{(z_\alpha - 1)^2} = -\frac{(N-1)(N-5)}{12} \quad , \tag{17}$$

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{(z_\alpha - 1)} = \frac{N-3}{2} \quad , \tag{18}$$

$$\sum_{\alpha=1}^{N-1} z_\alpha^2 = -1 \quad , \tag{19}$$

$$\sum_{\alpha=1}^{N-1} z_\alpha^2 (z_\alpha - 1) = \ldots = \sum_{\alpha=1}^{N-1} z_\alpha^2 (z_\alpha - 1)^{N-3} = 0 \quad , \tag{20}$$

$$\frac{z_\alpha^2}{(z_\alpha - z_\beta)(z_\alpha - z_\gamma)} + \frac{z_\beta^2}{(z_\beta - z_\alpha)(z_\beta - z_\gamma)} + \frac{z_\gamma^2}{(z_\gamma - z_\alpha)(z_\gamma - z_\beta)} = 1 \; , \tag{21}$$

which are worked out in Appendix A. We also have

$$[\left\{ \sum_{\beta \neq \alpha}^{N} \frac{S_\alpha^z S_\beta^z}{|\, z_\alpha - z_\beta \,|^2} \right\} \Psi](z_1, \ldots, z_{N/2})$$

$$= \left\{ -\frac{N(N^2-1)}{48} + \sum_{j \neq k}^{N/2} \frac{1}{|\, z_j - z_k \,|^2} \right\} \Psi(z_1, \ldots, z_{N/2}) \quad . \tag{22}$$

This completes the proof, since

$$\mathcal{H}_{HS} = \frac{1}{2} J (\frac{2\pi}{N})^2 \left\{ \sum_{\alpha \neq \beta} \frac{S_\alpha^+ S_\beta^-}{|\, z_\alpha - z_\beta \,|^2} + \sum_{\alpha \neq \beta} \frac{S_\alpha^z S_\beta^z}{|\, z_\alpha - z_\beta \,|^2} \right\} \quad . \ \Box \tag{23}$$

Important Properties

1. **Reality:** Since z_α lies on the unit circle we have

$$\Psi^*(z_1, \ldots, z_{N/2}) = \prod_{j<k}^{N/2} (z_j^* - z_k^*)^2 \prod_j^{N/2} z_j^*$$

$$= \prod_{j<k}^{N/2} (z_k - z_j)^2 \prod_j^{N/2} z_j^{1-N} = \Psi(z_1, \ldots, z_{N/2}) \quad . \tag{24}$$

   Thus Ψ is real despite being a polynomial in the complex variables z_j.

2. **Translational Invariance:** Ψ is translated one lattice spacing by multiplying each of its arguments by $z = \exp(i2\pi/N)$. Since it is a homogeneous polynomial of degree $(N/2)^2$ we have

$$\Psi(z_1 z, \ldots, z_{N/2} z) = \exp(iN\pi/2) \; \Psi(z_1, \ldots, z_{N/2}) \quad . \tag{25}$$

   The crystal momentum of the state, i.e. the phase it acquires under translation, is thus 0 or π, depending on the value of the even integer N.

3. **Spin Rotational Invariance:** To prove Ψ is a spin singlet it suffices to show that it is an eigenstate of S^z with eigenvalue zero and that it is destroyed by the spin lowering operator S^-. The former is true for any wavefunction in which the number of z_j is constrained to $N/2$. For the latter we have

$$[S^- \Psi](z_2, \ldots, z_{N/2}) = \sum_{\alpha=1}^{N} \Psi(z_\alpha, z_2, \ldots, z_{N/2})$$

$$= \lim_{z_1 \to 0} \sum_{\ell=1}^{N-1} \frac{1}{\ell!} \left\{ \sum_{\alpha=1}^{N} z_\alpha^\ell \right\} \frac{\partial^\ell}{\partial z_1^\ell} \Psi(z_1, z_2, \ldots, z_{N/2}) = 0 \quad , \tag{26}$$

since

$$\sum_{\alpha=1}^{N} z_\alpha^\ell = N \; \delta_{\ell 0} \quad (\text{mod } N) \quad . \tag{27}$$

This implies that the wavefunction is the same with the roles of $\uparrow$ and $\downarrow$ reversed or, more generally, with the quantization axis taken to be an arbitrary direction in spin space.

4. **Quantum Disorder:** In Fig. 2 we plot the spin-spin correlation function

$$< \vec{S}_\alpha \cdot \vec{S}_\beta >$$

$$= \frac{3}{2} (\frac{N}{2} - 1) \frac{\sum_{z_3, \ldots, z_{N/2}} |\Psi(z_\alpha, z_\beta, z_3, ..., z_{N/2})|^2}{\sum_{z_2, \ldots, z_{N/2}} |\Psi(z_\alpha, z_2, ..., z_{N/2})|^2} - \frac{3}{4} \quad . \tag{28}$$

The convergence of this function to zero as $|z_\alpha - z_\beta| \rightarrow \infty$ shows that $|\Psi>$ has no long-range order and is a spin liquid. The fall-off is slow, however, and this is important. Strongly-disordered spin liquids, i.e. with exponential decay of correlations on a length scale ξ, are easy to construct when the spin in the unit cell is integral. They have an energy gap $\Delta = \hbar v / \xi$, where v is the spin-wave velocity of a nearby ordered state. But strongly-disordered spin liquids *cannot* be stabilized with short-range interactions when the spin in the unit cell is half-integral. The excitation spectrum in this case is always gapless [6].

5. **Factorizability:** The fact that $|\Psi>$ is a product of pair factors makes a number of its properties easy to calculate by semi-classical monte-carlo techniques. Let us, for example, consider Eq. (28). Writing

$$|\Psi(z_1, ..., z_{N/2})|^2 = \exp\Bigg[-\phi(z_1, ..., z_{N/2}) \Bigg]$$

$$\phi(z_1, ..., z_{N/2}) = -4 \sum_{j<k} \ln |z_j - z_k| \quad , \tag{29}$$

we see that the summand is the Boltzmann factor of an equivalent finite-temperature classical lattice gas, and that we are computing the joint probability for two of these particles to reside at sites z_α and z_β. This may be done by generating a time sequence of configurations using a rule that obeys detailed balance, and then simply counting how many times the sites z_α and z_β are simultaneously occupied. The simplest such algorithm is the following. Let the current configuration be $z_1, ..., z_{N/2}$.

(a) Loop on particles j.

(b) For this particle, roll the dice to choose a direction. Compute

$$z_j' = z_j \exp(\pm i 2\pi/N) \tag{30}$$

depending on the outcome.

(c) Compute

$$f = \prod_{k \neq j} \left| \frac{z_j' - z_k}{z_j - z_k} \right|^4 \quad . \tag{31}$$

(d) If $f > 1$ update z_j to z_j'.

(e) If $f < 1$, roll the dice to generate a real number x between 0 and 1. Update z_j to z_j' if $x < f$ but do nothing otherwise.

Fig. 2 was generated using this algorithm.

6. **Degeneracy:** The Haldane-Shastry ground state is not degenerate, but it is nearly so. The alternate ground state is

$$\Psi'(z_1, ..., z_{N/2}) = \prod_{j<k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2 \right] \quad . \tag{32}$$

It has crystal momentum π greater than that of Ψ and has energy

$$\mathcal{H}_{HS} |\Psi'> = -J \left(\frac{\pi^2}{24} \right) \left(N - \frac{7}{N} \right) |\Psi'> \quad . \tag{33}$$

It is equivalent to the original vacuum plus a pair of spinons excited out of the vacuum into a singlet with total momentum π.

4 Spinons

Let the number of sites N be odd and let

$$\Psi_\alpha(z_1, ..., z_M) = \prod_j^M (z_\alpha - z_j) \prod_{j<k}^M (z_j - z_k)^2 \prod_j^M z_j \quad , \tag{34}$$

where $M = (N-1)/2$. This is a $\downarrow$ spin on site α surrounded by an otherwise featureless singlet sea. We have

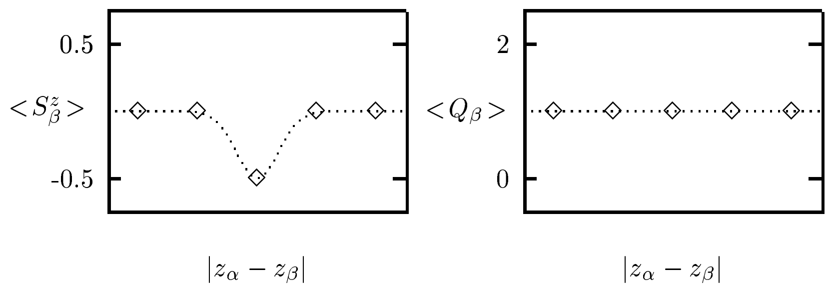

Figure 3: Spin and charge profiles of the localized spinon $|\Psi_\alpha>$ defined by Eq. (34). The dotted lines are a guide to the eye.

$$\sum_{\beta\neq\alpha}^{N} S_\beta^- \Psi_\alpha = 0 \quad , \tag{35}$$

per Eq. (26). The combination of these states given by

$$\Psi_m(z_1,\ldots,z_M) = \sum_{\alpha}^{N} (z_\alpha^*)^m \; \Psi_\alpha(z_1,\ldots,z_M) \quad , \tag{36}$$

with $0 \leq m \leq (N-1)/2$, is an exact eigenstate of the Hamiltonian with eigenvalue

$$\mathcal{H}_{HS}|\Psi_m> = \left\{ -J(\frac{\pi^2}{24})(N - \frac{1}{N}) + \frac{J}{2}(\frac{2\pi}{N})^2 m(\frac{N-1}{2} - m) \right\} |\Psi_m> \quad . \tag{37}$$

Proof

Following Haldane [1] we consider a wavefunction of the general form

$$\Psi(z_1,\ldots,z_M) = \Phi(z_1,\ldots,z_M) \; \prod_{j<k}^{M} (z_j - z_k)^2 \prod_{j}^{M} z_j \quad , \tag{38}$$

where $M = (N-1)/2$ and Φ is a homogeneous symmetric polynomial of degree less than $N - 2M + 2$. This latter condition causes Ψ to be a polynomial of degree less than $N + 1$ in each of its variables z_j, and thus allows the Taylor expansion technique used for the ground state to be applied. Doing so, we find that

$$\mathcal{H}_{HS}\Psi = \frac{J}{2}(\frac{2\pi}{N})^2 \left\{ \lambda + \frac{N}{48}(N^2-1) + \frac{M}{6}(4M^2-1) - \frac{N}{2}M^2 \right\} \Psi \quad , \tag{39}$$

provided that Φ satisfies

$$\frac{1}{2}\left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi}{\partial z_j^2} + \sum_{j\neq k}^M \frac{4z_j^2}{z_j - z_k}\frac{\partial \Phi}{\partial z_j} \right\} - \frac{N-3}{2}\sum_j^M z_j \frac{\partial \Phi}{\partial z_j} = \lambda \Phi \quad . \tag{40}$$

Let us now consider the polynomial

$$\Phi_A(z_1,\ldots,z_M) = \prod_j^M (z_A - z_j) = \sum_{m=0}^M z_A^m P_m(z_1,\ldots,z_M) \quad , \tag{41}$$

where z_A is not necessarily a lattice site. We have

$$\frac{1}{2}\left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi_A}{\partial z_j^2} + \sum_{j\neq k}^M \frac{4z_j^2}{z_j - z_k}\frac{\partial \Phi_A}{\partial z_j} \right\} - \frac{N-3}{2}\sum_j^M z_j \frac{\partial \Phi_A}{\partial z_j}$$

$$= 2\sum_{j<k}\left[\frac{z_j^2}{(z_j - z_A)(z_j - z_k)} + \frac{z_k^2}{(z_k - z_A)(z_k - z_j)} \right]\Phi_A$$

$$-\frac{N-3}{2}\sum_j^M \frac{z_j}{z_j - z_A}\Phi_A$$

$$= \left\{ M(M-1) - z_A^2\frac{\partial^2}{\partial z_A^2} - \frac{N-3}{2}\left[M - z_A \frac{\partial}{\partial z_A} \right] \right\}\Phi_A$$

$$= \sum_{m=0}^M m(\frac{N-1}{2} - m)\; z_A^m P_m \quad . \tag{42}$$

The proof is completed by multiplying both sides of this equation by $(z_A^*)^n$ and then summing on lattice sites z_A. $\Box$

The state $|\Psi_m>$ is a propagating $\downarrow$ spinon with crystal momentum

$$q = \frac{\pi}{2}N - \frac{2\pi}{N}(m + \frac{1}{4}) \pmod{2\pi} \quad , \tag{43}$$

per the definition

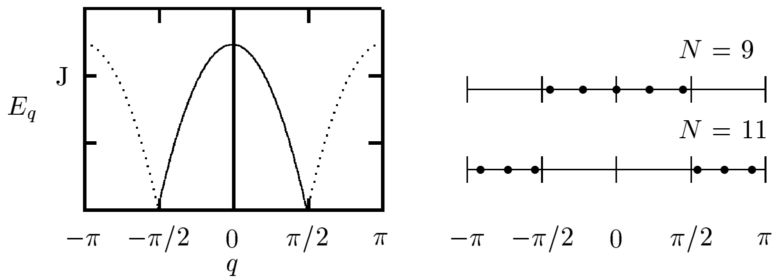

Figure 4: Left: Spinon dispersion given by Eq. (46). Right: Allowed values of q for adjacent odd N.

$$\Psi_m(z_1 z, \ldots, z_M z) = \exp(iq)\ \Psi_m(z_1, \ldots, z_M) \quad . \tag{44}$$

Rewriting the eigenvalue as

$$\mathcal{H}|\Psi_m\!> = \left\{ -J(\frac{\pi^2}{24})(N + \frac{5}{N} - \frac{3}{N^2}) + E_q \right\} |\Psi_m\!> \quad , \tag{45}$$

we obtain the dispersion relation

$$E_q = \frac{J}{2} \left[(\frac{\pi}{2})^2 - q^2 \right] \quad (\text{mod } \pi) \tag{46}$$

plotted in Fig. 4. Note that the momenta available to the spinon span only the inner or outer half of the Brillouin zone, depending on whether $N-1$ is divisible by 4. The spinon dispersion at low energies is linear in q with a velocity

$$v_{spinon} = \frac{\pi}{2} \frac{Jb}{\hbar} \quad , \tag{47}$$

where $b = 2\pi/N$ is the bond length.

The existence of spinons is an automatic consequence of quantum disorder whenever the total spin in the unit cell is half-integral. The spin liquid ground state must be a singlet because otherwise it is ferromagnetic. A singlet is possible only if the total number of lattice sites N is even. If N is odd then the total spin can be no less than 1/2. But since there can be no physical difference between even and odd in the limit of large N, the system must have had a neutral spin-1/2 excitation to begin with.

The ground state of the odd-N spin chain is 4-fold degenerate and is given by $|\Psi_m\!>$ for $m = 0$ and $(N-1)/2$ and their $\uparrow$ counterparts. This corresponds physically to a "left-over" spinon with momentum $\pm\pi$.

Spinons maintain their identity when more than one of them is present. When N is even, for example, the states

$$|\Psi_{mn}\!>= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\; f_{mn}(z_\alpha^* z_\beta)\; (z_\alpha^*)^m (z_\beta^*)^n |\Psi_{\alpha\beta}\!> \quad , \tag{48}$$

where

$$\Psi_{\alpha\beta}(z_1,\ldots,z_M) = \prod_j^M (z_\alpha - z_j)(z_\beta - z_j)\; \prod_{j<k}^M (z_j - z_k)^2 \prod_j^M z_j \quad , \tag{49}$$

with $M = N/2 - 1$ and

$$f_{mn}(z) = \sum_{\ell=0}^{N/2}\; a_\ell z^\ell - \frac{1}{2} a_{N/2} z^{N/2}$$

$$a_\ell = \frac{m - n + 2\ell}{2\ell[\ell + m - n - 1/2]} \sum_{k=0}^{\ell-1}\; a_k \qquad (a_0 = 1) \tag{50}$$

with $m \geq n$, are eigenstates of the Hamiltonian with eigenvalue

$$\mathcal{H}_{HS}|\Psi_{mn}\!>= \Bigg\{ -J(\frac{\pi^2}{24})(N - \frac{19}{N} + \frac{24}{N^2}) + \frac{J}{2}(\frac{2\pi}{N})^2$$

$$\times \left[m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) - \left| \frac{m-n}{2} \right| \right] \Bigg\} |\Psi_{mn}\!> \quad . \tag{51}$$

Proof

Following the procedure we use for one spinon we take Φ to be a superposition of states of the form

$$\Phi_{AB} = \prod_j^M (z_A - z_j)(z_B - z_j)$$

$$= \sum_{m=0}^{M}\; \sum_{n=0}^{M}\; z_A^m z_B^n\; P_m(z_1,\ldots,z_M)\; P_n(z_1,\ldots,z_M) \quad , \tag{52}$$

where z_A and z_B are not necessarily lattice sites. We find that

$$\frac{1}{2}\Bigg\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi_{AB}}{\partial z_j^2} + \sum_{j\neq k}^M \frac{4z_j^2}{z_j - z_k} \frac{\partial \Phi_{AB}}{\partial z_j} \Bigg\} - \frac{N-3}{2} \sum_j^M z_j \frac{\partial \Phi_{AB}}{\partial z_j}$$

$$= \Bigg\{ -\frac{z_A^2}{z_A - z_B}\frac{\partial}{\partial z_A} - \frac{z_B^2}{z_B - z_A}\frac{\partial}{\partial z_B} - z_A^2\frac{\partial^2}{\partial z_A^2} - z_B^2\frac{\partial^2}{\partial z_B^2}$$

$$+(\frac{N-3}{2})\left[z_A\frac{\partial}{\partial z_A} + z_B\frac{\partial}{\partial z_B}\right] + \left[2M^2 - M(N-2)\right]\Bigg\}\Phi_{AB}$$

$$= \sum_{m=0}^{M}\sum_{n=0}^{M}\Bigg\{ m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n)$$

$$- (\frac{m-n}{2})\frac{z_A + z_B}{z_A - z_B}\Bigg\} z_A^m z_B^n P_m P_n \quad , \tag{53}$$

and thus that

$$\frac{1}{2}\Bigg\{\sum_j^M z_j^2 \frac{\partial^2 \Phi_{mn}}{\partial z_j^2} + \sum_{j\neq k}^M \frac{4z_j^2}{z_j - z_k}\frac{\partial \Phi_{mn}}{\partial z_j}\Bigg\} - \frac{N-3}{2}\sum_j^M z_j \frac{\partial \Phi_{mn}}{\partial z_j}$$

$$= \Bigg\{ m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) + \frac{m-n}{2}\Bigg\}\Phi_{mn}$$

$$- \sum_{\ell=0}^{n}(m - n + 2\ell)\ \Phi_{m+\ell, n-\ell} \quad , \tag{54}$$

for $m \geq n$, where

$$\Phi_{mn} = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}(z_\alpha^*)^m (z_\beta^*)^n \Phi_{\alpha\beta} = N^2 P_m P_n \quad . \tag{55}$$

In obtaining this last expression we have used the identity

$$\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{\ell=0}^{m-n} x^{m-\ell} y^{n+\ell} - (x^m y^n + x^n y^m) \quad . \tag{56}$$

The solution of Eq. (40) is then

$$\Phi = \sum_{\ell=0}^{n} a_\ell\ \Phi_{m+\ell, n-\ell}$$

$$\lambda = m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) - \frac{m-n}{2} \quad , \tag{57}$$

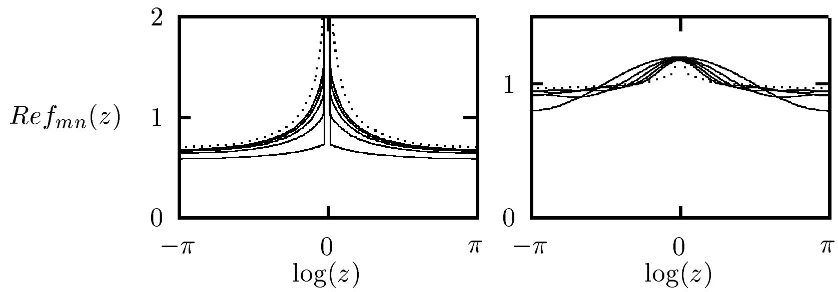

Figure 5: Left: Real part of spinon pair internal wavefunction $f_{mn}(z)$ defined by Eq. (50) for the case of $N = 100$ and $m - n = 2, 4, ..., 10$ (solid) and 50 (dots). Right: The "adjoint" function $\bar{f}_{mn}(z)$ defined in Problem 7.

where the coefficients a_ℓ are given by Eq. (50). Such a simple solution is possible because the matrix to which Eq. (54) corresponds is lower triangular, i.e. takes the form

$$\text{Matrix} = \begin{bmatrix} E_0 & 0 & 0 & 0 & \dots \\ v_{10} & E_1 & 0 & 0 & \\ v_{20} & v_{21} & E_2 & 0 & \\ v_{30} & v_{31} & v_{32} & E_3 & \dots \\ \vdots & & & \vdots & \end{bmatrix} \quad . \tag{58}$$

The eigenvalues of such a matrix are its diagonal elements, and the corresponding eigenvectors are generated by recursion. It should be noted that the upper bound on the sum in Eq. (50) is flexible, as Φ_{mn} is identically zero unless $0 \leq m, n \leq M \pmod{N}$. We have chosen the largest possible value so as to optimize the smoothness and short-rangeness of $f_{mn}(z)$. $\square$

It may be seen in Fig. 5 that $f_{mn}(z_\alpha/z_\beta)$ exhibits a scattering resonance, an enhancement when $z_\alpha^* z_\beta \simeq 1$, indicating that the spinons attract each other. This attractive force may also be inferred from the energy eigenvalue if we rewrite it as

$$\mathcal{H}_{HS}|\Psi_{mn}> = \left\{ -J(\frac{\pi^2}{24})(N + \frac{5}{N}) + E_{q_1} + E_{q_2} + V_{q_1 - q_2} \right\} |\Psi_{mn}> \quad , \tag{59}$$

where

$$q_1 = \frac{\pi}{2} - \frac{2\pi}{N}(m + \frac{1}{2}) \qquad q_2 = \frac{\pi}{2} - \frac{2\pi}{N}(n + \frac{1}{2}) \tag{60}$$

are the spinon momenta, E_q is defined as in Eq. (46), and

$$V_q = -J\frac{\pi}{N}\ |q| \quad . \tag{61}$$

Note that this potential vanishes as $N \to \infty$, as expected for particles that interact only when they are close together, and that the total crystal momentum, as defined by Eq. (44), is

$$q = \frac{\pi}{2}(N-2) + q_1 + q_2 \qquad (\mathrm{mod}\ 2\pi) \quad , \tag{62}$$

a value greater by π than that of the ground state when $q_1 = q_2 = 0$.

For every triplet state $|\Psi_{mn}>$ defined as in Eq. (48) there is a corresponding singlet $\Lambda^z S^+|\Psi_{mn}>$, where $\vec{\Lambda}$ is defined as in Eq. (4), with exactly the same energy eigenvalue. For example, for the case of $m = N/2-1$ and $n = 0$ we have

$$[\Lambda^z S^+ \Psi_{mn}](z_1, ..., z_{N/2}) = -N^2 \prod_{j<k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2\right] \quad , \tag{63}$$

which may be seen to be the alternate ground state defined in Eq. (32).

Proof

We begin by using the sum rules

$$\prod_j^M z_j \prod_j^{N-M} \eta_j = 1 \qquad\qquad z_j \prod_{k \neq j}^M (z_j - z_k) \prod_k^{N-M} (z_j - \eta_k) = N \quad , \tag{64}$$

to rewrite the wavefunction in terms of the $\downarrow$ spin locations η_j, per

$$\Psi_{mn}(z_1, ..., z_{N/2-1}) = \prod_{j<k}^{N/2-1} (z_j - z_k)^2 \prod_j^{N/2-1} z_j^2 = \prod_{j<k}^{N/2+1} (\eta_j - \eta_k)^2 \quad . \tag{65}$$

We then have

$$[S^+\Psi_{mn}](z_1, ..., z_{N/2}) = \sum_\alpha \Psi_{mn}(\eta_1, ..., \eta_{N/2}, z_\alpha)$$

$$= N \prod_{j<k}^{N/2} (\eta_j - \eta_k)^2 \left[1 + \prod_{j=1}^{N/2} \eta_j^2\right] = N \prod_{j<k}^{N/2} (z_j - z_k)^2 \left[1 + \prod_{j=1}^{N/2} z_j^2\right] \quad , \tag{66}$$

and thus

$$[\Lambda^z S^+ \Psi_{mn}](z_1, ..., z_{N/2})$$

$$= \frac{1}{2} \sum_{\alpha \neq \beta} (\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta}) [S_\alpha^+ S_\beta^- S^+ \Psi_{mn}](z_1, ..., z_{N/2})$$

$$= \frac{1}{2} \sum_j \sum_{\beta \neq j} (\frac{z_j + z_\beta}{z_j - z_\beta}) [S^+ \Psi_{mn}](z_1, ..., z_{j-1}, z_\beta, z_{j+1}, ..., z_{N/2})$$

$$= N \sum_j^{N/2} z_j \frac{\partial}{\partial z_j} \left\{ \prod_{j<k}^{N/2} (z_j - z_k)^2 \left[1 + \prod_{j=1}^{N/2} z_j^2 \right] \right\}$$

$$= -N^2 \prod_{j<k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2 \right] \quad . \square \tag{67}$$

The singlet has no simple exact representation in terms of $|\Psi_{\alpha\beta}>$ but is reasonably approximated by

$$\Lambda^z S^+ |\Psi_{mn}>$$

$$\simeq \frac{1}{2} \sum_{\alpha \neq \beta}^{N} f_{mn}(z_\alpha / z_\beta) \; (z_\alpha^*)^m (z_\beta^*)^n (\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta}) (S_\alpha^+ - S_\beta^+) |\Psi_{\alpha\beta}> \quad . \tag{68}$$

That it is an energy eigenstate follows from the conservation of $\vec{\Lambda}$ and the non-commutativity of $\vec{\Lambda}$ and $\vec{S}$ implicit in Eqs. (11). That it is a singlet follows from

$$\Lambda^z |\Psi_{mn}> = \left\{ \frac{N-2}{2} - m - n \right\} |\Psi_{mn}> \quad , \tag{69}$$

i.e. that $|\Psi_{mn}>$ is an eigenstate of Λ^z, an important result we shall revisit. This implies that the spin-2 representation contained in the 9 states $\Lambda^\mu S^\nu |\Psi_{mn}>$ $(\mu, \nu = 1, 2, 3)$ must be identically zero and the spin-1 representation must be just $|\Psi_{mn}>$ itself.

Spinons are semions, i.e. particles obeying 1/2 fractional statistics. Since the 2-spinon wavefunction

$$\Psi_{AB} = \prod_j (z_j - z_A)(z_j - z_B) \prod_{j<k} (z_j - z_k)^2 \quad , \tag{70}$$

where z_A and z_B are not necessarily lattice sites, has the property

$$\Psi_{AB}^*(z_1, ..., z_{N/2-1}) = (z_A z_B)^{1-N/2} \Psi_{AB}(z_1, ..., z_{N/2-1}) \quad , \tag{71}$$

the Berry phase vector potential for adiabatic motion of spinon A in the presence of B is

$$\frac{1}{2}\Bigg[< \psi_{AB} | z_A \frac{\partial}{\partial z_A} \psi_{AB} > + < z_A \frac{\partial}{\partial z_A} \psi_{AB} | \psi_{AB} > \Bigg] / < \psi_{AB} | \psi_{AB} >$$

$$= \frac{1}{2}(1 - \frac{N}{2}) \quad . \tag{72}$$

The phase to "exchange" the spinons by moving A all the way around the loop is thus

$$\Delta\phi = \oint \frac{1}{2}(1 - \frac{N}{2}) \frac{dz_A}{z_A} = \pm \frac{\pi}{2} i \qquad (\mathrm{mod}\ 2\pi) \quad . \tag{73}$$

This number is 0 or π for bosons or fermions. Fractional statistics is actually the long-range force between the spinons manifested in the resonant enhancements of Fig. 5 and the potential $V_{q_1 - q_2}$ in Eq. (59), and has nothing to do with the symmetry or antisymmetry of $|\Psi_{\alpha\beta}>$ under interchange of α and β. It does have to do with state-counting. The number of states available to $\ell \downarrow$ spinons, determined by counting the number of distinct symmetric polynomials of the form

$$\Phi_{z_{A_1}, ..., z_{A_\ell}}(z_1, ..., z_{(N-\ell)/2}) = \prod_j^{(N-\ell)/2} (z_j - z_{A_1}) \times ... \times (z_j - z_{A_\ell}) \quad , \tag{74}$$

is

$$\mathcal{N}_\ell^{\mathrm{semi}} = \binom{N/2 + \ell/2}{\ell} \quad . \tag{75}$$

This is just halfway between the numbers

$$\mathcal{N}_\ell^{\mathrm{fermi}} = \binom{N/2}{\ell} \qquad\qquad \mathcal{N}_\ell^{\mathrm{bose}} = \binom{N/2 + \ell}{\ell} \quad , \tag{76}$$

likewise calculated assuming that the number of states available for one particle is $N/2$.

5 Annihilation Operators

The operators

$$\vec{\Omega}_\alpha = \frac{1}{2} \sum_{\beta \neq \alpha} \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta] \tag{77}$$

annihilate the Haldane-Shastry ground state, i.e. satisfy

$$\vec{\Omega}_\alpha |\Psi >= 0 \quad , \tag{78}$$

for all α.

Proof

We have as before that $[S_\alpha^+ S_\beta^- \Psi](z_1, ..., z_{N/2})$ is zero unless one of the arguments $z_1, ..., z_{N/2}$ equals z_α, but in this case we do not sum over z_α. Instead we have

$$\sum_{\beta \neq \alpha}^{N} \frac{z_\alpha}{z_\alpha - z_\beta} [S_\alpha^+ S_\beta^- \Psi](z_\alpha, z_2, ..., z_{N/2}) = \sum_{\beta \neq \alpha}^{N} \frac{z_\alpha}{z_\alpha - z_\beta} \Psi(z_\beta, z_2, ..., z_{N/2})$$

$$= \sum_{\ell=0}^{N-2} \left\{ \frac{1}{\ell!} \sum_{\beta \neq \alpha}^{N} \frac{z_\alpha z_\beta (z_\beta - z_\alpha)^\ell}{z_\alpha - z_\beta} \right\} \frac{\partial^\ell}{\partial z_\alpha^\ell} \left\{ \frac{\Psi(z_\alpha, z_2, ..., z_{N/2})}{z_\alpha} \right\}$$

$$= \left\{ -\frac{N-1}{2} + 2 \sum_{j \neq \alpha}^{N/2} \frac{z_\alpha}{z_\alpha - z_j} \right\} \Psi(z_\alpha, z_2, ..., z_{N/2}) \quad . \tag{79}$$

However $(1/2 + S_\alpha^z)|\Psi >$ is also identically zero unless one of the arguments $z_1, ..., z_{N/2}$ equals z_α. We thus have

$$\sum_{\beta \neq \alpha}^{N} [\frac{z_\alpha}{z_\alpha - z_\beta} (\frac{1}{2} + S_\alpha^z)(\frac{1}{2} + S_\beta^z) \Psi](z_\alpha, z_2, ..., z_{N/2})$$

$$= \sum_{j \neq \alpha} \frac{z_\alpha}{z_\alpha - z_j} \Psi(z_\alpha, z_2, ..., z_{N/2}) \quad . \tag{80}$$

Subtracting these from each other we find that

$$\left\{ \sum_{\beta \neq \alpha}^{N} \frac{z_\alpha}{z_\alpha - z_\beta} \left[S_\alpha^+ S_\beta^- - 2(\frac{1}{2} + S_\alpha^z)(\frac{1}{2} + S_\beta^z) \right] \right.$$

$$+\frac{N-1}{2}(\frac{1}{2}+S_\alpha^z)\Big\}|\Psi>=0 \qquad (81)$$

for all α. However since $|\Psi>$ is a spin singlet the irreducible representations of the rotation group present in this operator must destroy $|\Psi>$ separately. The scalar component is identically zero. The vector component is

$$\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha}{z_\alpha-z_\beta}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]|\Psi>=0 \quad . \qquad (82)$$

The rank-2 tensor component is the product of the two and therefore also zero. Since $|\Psi>$ is also its own time-reverse it must be destroyed by the time-reverse of the vector operator, i.e.

$$\sum_{\beta\neq\alpha}\frac{z_\alpha^*}{z_\alpha^*-z_\beta^*}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]=-\sum_{\beta\neq\alpha}\frac{z_\beta}{z_\alpha-z_\beta}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta] \quad . \qquad (83)$$

The difference of these is the trivial operator $\vec{S}_\alpha\times\vec{S}$, and their sum is $2\vec{\Omega}_\alpha$. □

These operators satisfy

$$\vec{S}_\alpha\times\vec{\Omega}_\alpha=-\frac{i}{2}\vec{\Omega}_\alpha \qquad \vec{S}_\alpha\cdot\vec{\Omega}_\alpha=0 \qquad (84)$$

and are related to $\vec{\Lambda}$ by

$$\sum_\alpha\vec{\Omega}_\alpha=\vec{\Lambda} \quad . \qquad (85)$$

They are not symmetries of the Hamiltonian but supercharges, for we have

$$\sum_\alpha\vec{\Omega}_\alpha^\dagger\cdot\vec{\Omega}_\alpha$$

$$=\frac{3}{2}\left[3\sum_{\alpha\neq\beta}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}+\frac{N(N^2+5)}{16}-\frac{(N+1)}{4}S^2\right] \quad , \qquad (86)$$

exactly.

Proof

Since

$$\vec{\Omega}_\alpha = \sum_{\beta \neq \alpha} \frac{z_\alpha}{z_\alpha - z_\beta} [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta] - \frac{1}{2}[i(\vec{S}_\alpha \times \vec{S}) + \vec{S}] \quad , \tag{87}$$

per the previous discussion, we have

$$\sum_\alpha \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \frac{[i(\vec{S}_\alpha \times \vec{S}_\gamma) + \vec{S}_\gamma]^\dagger \cdot [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta]}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)}$$

$$= \sum_\alpha \vec{\Omega}_\alpha^\dagger \cdot \vec{\Omega}_\alpha + \frac{3}{2}\vec{S} \cdot \vec{\Lambda} + \frac{3}{8}(N-1)S^2 \quad . \tag{88}$$

In evaluating this expression we have used

$$[i(\vec{S}_\alpha \times \vec{S}_\gamma) + \vec{S}_\gamma]^\dagger = [i(\vec{S}_\gamma \times \vec{S}_\alpha) + \vec{S}_\gamma] \quad , \tag{89}$$

$$\sum_\alpha [i(\vec{S} \times \vec{S}_\alpha) + \vec{S}] \cdot \vec{\Omega}_\alpha = \sum_\alpha [i\vec{S} \cdot (\vec{S}_\alpha \times \vec{\Omega}_\alpha) + \vec{S} \cdot \vec{\Omega}_\alpha] = \frac{3}{2}\vec{S} \cdot \vec{\Lambda} \quad , \tag{90}$$

and

$$\sum_\alpha [i(\vec{S} \times \vec{S}_\alpha) + \vec{S}] \cdot [i(\vec{S}_\alpha \times \vec{S}) + \vec{S}]$$

$$= \frac{3}{2} \sum_\alpha [S^2 - i\vec{S} \cdot (\vec{S}_\alpha \times \vec{S})] = \frac{3}{2}[N-1]S^2 \quad , \tag{91}$$

which follows from

$$[i(\vec{S}_\gamma \times \vec{S}_\alpha) + \vec{S}_\gamma] \cdot [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta]$$

$$= -(\vec{S}_\gamma \times \vec{S}_\alpha) \cdot (\vec{S}_\alpha \times \vec{S}_\beta) + i\vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta) + i(\vec{S}_\gamma \times \vec{S}_\alpha) \cdot \vec{S}_\beta + \vec{S}_\gamma \cdot \vec{S}_\beta$$

$$= (1 + \frac{3}{4})(\vec{S}_\gamma \cdot \vec{S}_\beta) - (\vec{S}_\gamma \cdot \vec{S}_\alpha)(\vec{S}_\alpha \cdot \vec{S}_\beta) + 2i\vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta)$$

$$= \frac{3}{2}[\vec{S}_\gamma \cdot \vec{S}_\beta + i\vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta)] \quad . \tag{92}$$

The 2-spin sum is

$$\sum_{\beta \neq \gamma \neq \alpha} \frac{\vec{S}_\beta \cdot \vec{S}_\gamma}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} = - \sum_{\alpha \neq \beta \neq \gamma} \frac{z_\alpha z_\gamma}{(z_\alpha - z_\gamma)(z_\alpha - z_\beta)} \vec{S}_\gamma \cdot \vec{S}_\beta$$

$$= -\sum_{\beta\neq\gamma} \frac{z_\gamma}{z_\beta - z_\gamma} \vec{S}_\gamma \cdot \vec{S}_\beta \sum_{\alpha\neq\beta,\gamma} \left[\frac{z_\alpha}{z_\alpha - z_\beta} - \frac{z_\alpha}{z_\alpha - z_\gamma} \right]$$

$$= -\sum_{\beta\neq\gamma} \frac{z_\gamma}{z_\beta - z_\gamma} \vec{S}_\gamma \cdot \vec{S}_\beta \left[\frac{z_\beta}{z_\beta - z_\gamma} - \frac{z_\gamma}{z_\gamma - z_\beta} \right]$$

$$= -\sum_{\beta\neq\gamma} \frac{z_\gamma (z_\gamma + z_\beta)}{(z_\beta - z_\gamma)^2} \vec{S}_\gamma \cdot \vec{S}_\beta = -\frac{1}{2} \sum_{\beta\neq\gamma} \left(\frac{z_\beta + z_\gamma}{z_\beta - z_\gamma} \right)^2 \vec{S}_\beta \cdot \vec{S}_\gamma$$

$$= -\frac{1}{2} S^2 + \frac{3}{8} N + 2 \sum_{\alpha\neq\beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} \quad . \tag{93}$$

The 3-spin sum is

$$i \sum_{\alpha\neq\beta\neq\gamma} \frac{\vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta)}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)}$$

$$= i \sum_{\alpha\neq\beta\neq\gamma} \frac{z_\alpha z_\gamma}{(z_\alpha - z_\gamma)(z_\alpha - z_\beta)} \vec{S}_\alpha \cdot (\vec{S}_\gamma \times \vec{S}_\beta)$$

$$= \frac{i}{2} \sum_{\alpha\neq\beta\neq\gamma} \frac{z_\alpha (z_\gamma - z_\beta)}{(z_\alpha - z_\beta)(z_\alpha - z_\gamma)} \vec{S}_\alpha \cdot (\vec{S}_\gamma \times \vec{S}_\beta)$$

$$= \frac{i}{2} \sum_{\alpha\neq\beta\neq\gamma} \left[\frac{z_\alpha}{z_\alpha - z_\gamma} - \frac{z_\alpha}{z_\alpha - z_\beta} \right] \vec{S}_\alpha \cdot (\vec{S}_\gamma \times \vec{S}_\beta)$$

$$= \frac{i}{2} \sum_{\alpha\neq\beta\neq\gamma} \left(\frac{z_\alpha + z_\gamma}{z_\alpha - z_\gamma} \right) (\vec{S}_\alpha \times \vec{S}_\gamma) \cdot \vec{S}_\beta = \vec{\Lambda} \cdot \vec{S} \quad . \tag{94}$$

Note that in the last step we have used the identity

$$(\vec{S}_\alpha \times \vec{S}_\gamma) \cdot (\vec{S}_\alpha + \vec{S}_\gamma) = 0 \quad . \tag{95}$$

Putting these results together, we find that

$$\sum_\alpha \sum_{\beta\neq\alpha} \sum_{\gamma\neq\alpha} \frac{1}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} [-i(\vec{S}_\alpha \times \vec{S}_\gamma) + \vec{S}_\gamma] \cdot [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta]$$

$$= \frac{3}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \frac{1}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} [\vec{S}_\gamma \cdot \vec{S}_\beta + i \vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta)]$$

$$= \frac{3}{2} \left\{ \sum_{\beta \neq \alpha} \frac{1}{|z_\alpha - z_\beta|^2} \left[\frac{3}{4} + 3\, \vec{S}_\alpha \cdot \vec{S}_\beta \right] - \frac{S^2}{2} + \frac{3}{8} N + \vec{\Lambda} \cdot \vec{S} \right\}$$

$$= \frac{3}{2} \left[3 \sum_{\alpha \neq \beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{N(N^2+5)}{16} - \frac{S^2}{2} + \vec{S} \cdot \vec{\Lambda} \right] \quad . \ \square \tag{96}$$

Since $<\Phi|\vec{\Omega}_\alpha^\dagger \cdot \vec{\Omega}_\alpha|\Phi>$ is non-negative for any wavefunction $|\Phi>$, this provides an explicit demonstration that $|\Psi>$ is the true ground state. The annihilation operators and their equivalence to $\mathcal{H}_{HS}$ when squared and summed were originally discovered by Shastry [7]. They are modeled after a similar set of operators discovered for the 2-dimensional chiral spin liquid, although there was a minus-sign error in the original paper which caused the operators to be mistakenly reported as scalars [8]. They are lattice versions of the Knizhnik-Zamolodchikov operators known from studies of the Cologero-Sutherland model, the 1-dimensional Bose gas with inverse-square repulsions [9, 10].

6 Spin Current

The operator $\vec{\Lambda}$ is a scaled spin current. Its action on the propagating spinon of Eq. (36), for example, is

$$\Lambda^z |\Psi_m> = \left\{ \frac{N-1}{4} - m \right\} |\Psi_m> \quad , \tag{97}$$

which is proportional to the spinon velocity

$$\frac{dE_q}{dq} = \frac{2\pi J}{N} \left\{ \frac{N-1}{4} - m \right\} \quad . \tag{98}$$

Its action on the 2-spinon state given by Eq. (69) is similarly the sum of the two spinon velocities.

Proof

We have, with $M = (N-1)/2$,

$$[\Lambda^z \Psi_A](z_1, ..., z_M) = \frac{1}{2} \sum_j^M \sum_{\beta \neq j} (\frac{z_j + z_\beta}{z_j - z_\beta}) \Psi_A(z_1, ..., z_{j-1}, z_\beta, z_{j+1}, ..., z_M)$$

$$= \frac{1}{2} \sum_j^M \sum_\ell \frac{1}{\ell!} \Big[\sum_{\beta \neq j} (\frac{z_j + z_\beta}{z_j - z_\beta}) (z_\beta - z_j)^\ell z_\beta \Big] \frac{\partial^\ell}{\partial z_j^\ell} \Big\{ \Psi_A(z_1, ..., z_M)/z_j \Big\}$$

$$= \Big\{ \frac{N-1}{4} - z_A \frac{\partial}{\partial z_A} \Big\} \Psi_A(z_1, ..., z_M) \quad , \tag{99}$$

and thus

$$\Lambda^z |\Psi_m >= \sum_{z_A} (z_A^*)^m \Lambda^z |\Psi_A >= \Big\{ \frac{N-1}{4} - m \Big\} |\Psi_m > \quad . \ \square \tag{100}$$

A more traditional description of this current may be constructed by interpolating the spin operators into the interstices by means of the formula

$$\vec{\sigma}(z) = \Big[\frac{z^{N/2} - z^{-N/2}}{2N} \Big] \sum_\beta \Big(\frac{z + z_\beta}{z - z_\beta} \Big) \vec{S}_\beta \quad . \tag{101}$$

The Hamiltonian is then

$$\frac{1}{2\pi i} \oint [z \frac{d\vec{\sigma}}{dz}] \cdot [z \frac{d\vec{\sigma}}{dz}] \frac{dz}{z} = -\frac{2}{N} \sum_{\alpha \neq \beta}^N \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{3}{8}(N-1) + \frac{S^2}{8} \quad , \tag{102}$$

where the integral is performed over the unit circle. We also have spin density and spin current density operators

$$\vec{\rho}(z) = -i \vec{\sigma}(z) \times \vec{\sigma}(z)$$

$$\vec{j}(z) = \frac{1}{2i} \Big\{ \vec{\sigma} \times [z \frac{d\vec{\sigma}}{dz}] - [z \frac{d\vec{\sigma}}{dz}] \times \vec{\sigma} \Big\} \quad , \tag{103}$$

which satisfy the continuity equation

$$\lim_{z \to z_\alpha} \Big\{ z \frac{d\vec{j}}{dz} + [\sum_{\alpha \neq \beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2}, \vec{\rho}] \Big\} = 0 \quad . \tag{104}$$

The zero-momentum component of this conserved current density is

$$\frac{1}{2\pi i} \oint \vec{j} \, \frac{dz}{z} = \vec{\Lambda} \quad . \tag{105}$$

7 Supersymmetry

We shall now consider the generalization of the Haldane-Shastry Hamiltonian

$$\mathcal{H}_{KY} = J \left(\frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta}^{N} \frac{1}{|z_\alpha - z_\beta|^2} \; P \Big\{ \vec{S}_\alpha \cdot \vec{S}_\beta$$

$$- \frac{1}{4} \sum_s (c_{\alpha s}^\dagger c_{\beta s} + c_{\beta s}^\dagger c_{\alpha s}) + \frac{1}{2}(n_\alpha + n_\beta) - \frac{1}{4} n_\alpha n_\beta - \frac{3}{4} \Big\} P \qquad (106)$$

first studied by Kuramoto and Yokoyama [11], where

$$P = \prod_\alpha \, (1 - c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow} c_{\alpha\uparrow}) \quad , \qquad (107)$$

is the Gutzwiller projector, and site occupation and spin operators are

$$n_\alpha = c_{\alpha\uparrow}^\dagger c_{\alpha\uparrow} + c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow} \qquad S_\alpha^x = \frac{1}{2}(c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow} + c_{\alpha\downarrow}^\dagger c_{\alpha\uparrow})$$

$$S_\alpha^y = \frac{1}{2i}(c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow} - c_{\alpha\downarrow}^\dagger c_{\alpha\uparrow}) \qquad S_\alpha^z = \frac{1}{2}(c_{\alpha\uparrow}^\dagger c_{\alpha\uparrow} - c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow}) \quad . \qquad (108)$$

Thus each site can be $\uparrow$, $\downarrow$, or unoccupied - but not doubly occupied - and the hole can tunnel to nearby sites by means of the same inverse-square matrix element characterizing the spin exchange. This equivalence of the energy scales for magnetism and charge transport causes the Hamiltonian to be supersymmetric in the sense of Eq. (86) and also in the more traditional one of commuting with electron or hole injection per

$$\sum_\alpha \, [\mathcal{H}_{KY}, P c_{\alpha s} P] = 0 \quad . \qquad (109)$$

Proof

It suffices to show that

$$[\mathcal{H}_{\alpha\beta}, (c_{\alpha\uparrow} + c_{\beta\uparrow})] \; |\psi\!>= 0 \quad , \qquad (110)$$

where

$$\mathcal{H}_{\alpha\beta} = P \Big\{ \vec{S}_\alpha \cdot \vec{S}_\beta - \frac{1}{4} \sum_s (c_{\alpha s}^\dagger c_{\beta s} + c_{\beta s}^\dagger c_{\alpha s})$$

$$- \frac{1}{4} n_\alpha n_\beta + \frac{1}{2}(n_\alpha + n_\beta - 1) \Big\} P \qquad (111)$$

for the 9 configurations $|\psi>$ on sites α and β allowed by the projector P. Denoting the state with no electron on either site by $|0\!>$ we have

- **Case 1:**

$$|\psi_1\!>=c^\dagger_{\alpha\uparrow}c^\dagger_{\beta\uparrow}|0\!> \tag{112}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_1\!>=\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_1\!>=\frac{1}{2}(c^\dagger_{\beta\uparrow}-c^\dagger_{\alpha\uparrow})|0\!>$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_1\!>=\mathcal{H}_{\alpha\beta}(c^\dagger_{\beta\uparrow}-c^\dagger_{\alpha\uparrow})|0\!>=\frac{1}{2}(c^\dagger_{\beta\uparrow}-c^\dagger_{\alpha\uparrow})|0\!>$$

- **Case 2:**

$$|\psi_2\!>=(c^\dagger_{\alpha\uparrow}c^\dagger_{\beta\downarrow}-c^\dagger_{\alpha\downarrow}c^\dagger_{\beta\uparrow})|0\!> \tag{113}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_2\!>=-\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_2\!>=-\frac{1}{2}(c^\dagger_{\beta\downarrow}+c^\dagger_{\alpha\downarrow})|0\!>$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_2\!>=\mathcal{H}_{\alpha\beta}(c^\dagger_{\beta\downarrow}+c^\dagger_{\alpha\downarrow})|0\!>=-\frac{1}{2}(c^\dagger_{\beta\downarrow}+c^\dagger_{\alpha\downarrow})|0\!>$$

- **Case 3:**

$$|\psi_3\!>=(c^\dagger_{\alpha\uparrow}c^\dagger_{\beta\downarrow}+c^\dagger_{\alpha\downarrow}c^\dagger_{\beta\uparrow})|0\!> \tag{114}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_3\!>=\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_3\!>=\frac{1}{2}(c^\dagger_{\beta\downarrow}-c^\dagger_{\alpha\downarrow})|0\!>$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_3\!>=\mathcal{H}_{\alpha\beta}(c^\dagger_{\beta\downarrow}-c^\dagger_{\alpha\downarrow})|0\!>=\frac{1}{2}(c^\dagger_{\beta\downarrow}-c^\dagger_{\alpha\downarrow})|0\!>$$

- **Case 4:**

$$|\psi_4\!>=c^\dagger_{\alpha\uparrow}c^\dagger_{\beta\uparrow}|0\!> \tag{115}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_4\!>=\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_4\!>=0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_4\!>=0$$

- **Cases 5 and 6:**

$$|\psi_5\!>=c_{\alpha\uparrow}|0\!> \tag{116}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_5\!>=-\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})c_{\beta\uparrow}^{\dagger}|0\!>=-\frac{1}{2}|0\!>$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_5\!>=\mathcal{H}_{\alpha\beta}|0\!>=-\frac{1}{2}|0\!>$$

- **Cases 7 and 8:**

$$|\psi_7\!>=c_{\alpha\downarrow}|0\!> \tag{117}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_7\!>=-\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})c_{\beta\downarrow}^{\dagger}|0\!>=0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_7\!>=0$$

- **Case 9:**

$$|\psi_9\!>=|0\!> \tag{118}$$

$$(c_{\alpha\uparrow}+c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_9\!>=-\frac{1}{2}(c_{\alpha\uparrow}+c_{\beta\uparrow})|0\!>=0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow}+c_{\beta\uparrow})|\psi_9\!>=0 \quad . \quad \Box$$

8 Holons

Holons are charged, spin-0 elementary excitations of the Kuramoto-Yokoyama Hamiltonian made by removing an electron from the center of a spinon. The localized holon $c_{a\downarrow}|\Psi_\alpha\!>$, where $|\Psi_\alpha\!>$ is defined as in Eq. (34), is shown in Fig. 6. It is a natural complement to the spinon from which it was made, for the two states differ only in the disposition of the central site, which does not fluctuate and is not involved in any way in the quantum number fractionalization. This

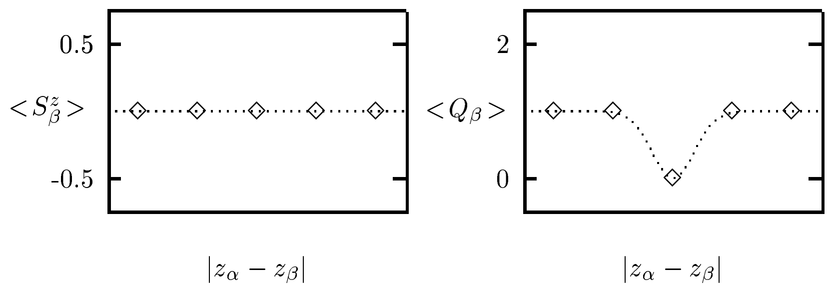

Figure 6: Spin and charge profiles of the localized holon $c_{a\downarrow}|\Psi_\alpha>$. The dotted lines are a guide to the eye.

state is an exact spin singlet by virtue of Eq. (35). Let N be odd and let $M = (N-1)/2$ as in Eq. (36). Then the propagating holon wavefunction

$$\Psi_m^{holon}(z_1, ..., z_M | h) = (h^*)^m \prod_j^M (h - z_j) \prod_{j<k}^M (z_j - z_k)^2 \prod_j^M z_j \quad , \tag{119}$$

where $z_1, ..., z_M$ denote the positions of the $\uparrow$ sites and h denotes the position of the empty site, all others being $\downarrow$, satisfies

$$\mathcal{H}_{KY} | \Psi_m^{holon} >$$

$$= \left\{ -J(\frac{\pi^2}{24})(N - \frac{1}{N}) + \frac{J}{2}(\frac{2\pi}{N})^2 m (\frac{N+1}{2} + m) \right\} |\Psi_m^{holon}> \quad , \tag{120}$$

provided that $-(N+1)/2 \leq m \leq 0$.

Proof

To simplify the notation let us negate the power of h in Eq. (119) so that the holon wavefunction becomes

$$\Psi_n(z_1, ..., z_M | h) = h^n \prod_j^M (h - z_j) \prod_{j<k}^M (z_j - z_k)^2 \prod_j^M z_j \quad , \tag{121}$$

with $0 \leq n \leq (N+1)/2$ and $M = (N-1)/2$. The effect of the spin-exchange part of the Hamiltonian is the same as for the spinon. Taylor expanding as usual we obtain

$$\sum_{\alpha\neq\beta}^{N}[\frac{S_\alpha^+ S_\beta^-}{|z_\alpha - z_\beta|^2}\Psi_n](z_1,...,z_M|h) = \Bigg\{ \Bigg[\frac{1-N^2}{24} - \sum_{j\neq k}^{M} \frac{1}{|z_j - z_k|^2} \Bigg] \; h^n$$

$$+ \frac{N-3}{2} h^{n+1} \frac{\partial}{\partial h} - h^{n+2} \frac{\partial^2}{\partial h^2} \Bigg\} \Bigg\{ \frac{\Psi_n(z_1,...,z_M|h)}{h^n} \Bigg\} \quad . \tag{122}$$

The charge-exchange terms also behave similarly. The operator $c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger$ gives zero unless the holon resides at z_α. For this case we have

$$\sum_{\beta\neq\alpha}^{N}[\frac{P c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger P}{|z_\alpha - z_\beta|^2}\Psi_n](z_1,...,z_M|z_\alpha) = \sum_{\beta\neq\alpha}^{N} \frac{1}{|z_\alpha - z_\beta|^2} \Psi_n(z_1,...,z_M|z_\beta)$$

$$= \sum_{\beta\neq\alpha}^{N} \sum_{\ell=0}^{M+1} \frac{z_\beta^n (z_\beta - z_\alpha)^\ell}{\ell! |z_\beta - z_\alpha|^2} (\frac{\partial}{\partial z_\alpha})^\ell \Bigg\{ \frac{\Psi_n(z_1,...,z_M|z_\alpha)}{z_\alpha^n} \Bigg\}$$

$$= \Bigg\{ \Bigg[\frac{N^2-1}{12} + \frac{n(n-N)}{2} \Bigg] z_\alpha^n$$

$$- \Bigg[\frac{N-1}{2} - n \Bigg] z_\alpha^{n+1} \frac{\partial}{\partial z_\alpha} + \frac{1}{2} z_\alpha^{n+2} \frac{\partial^2}{\partial z_\alpha^2} \Bigg\} \Bigg\{ \frac{\Psi_n(z_1,...,z_M|z_\alpha)}{z_\alpha^n} \Bigg\} \quad . \tag{123}$$

Summing on α we then obtain

$$\sum_{\alpha\neq\beta}^{N}[\frac{P c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger P}{|z_\alpha - z_\beta|}\Psi_n](z_1,...,z_M|h) = \Bigg\{ \Bigg[\frac{N^2-1}{12} + \frac{n(n-N)}{2} \Bigg] h^n$$

$$- \Bigg[\frac{N-1}{2} - n \Bigg] h^{n+1} \frac{\partial}{\partial h} + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \Bigg\} \Bigg\{ \frac{\Psi_n(z_1,...,z_M|h)}{h^n} \Bigg\} \quad . \tag{124}$$

For the other charge-exchange channel we use the fact that $|\Psi_n>$ is the same written in the $\downarrow$ coordinates $\eta_1, ..., \eta_M$ by virtue of being a singlet. This gives

$$\sum_{\alpha\neq\beta}^{N}[\frac{P c_{\alpha\uparrow} c_{\beta\uparrow}^\dagger P}{|z_\alpha - z_\beta|^2}\Psi_n](\eta_1,...,\eta_M|h) = \Bigg\{ \Bigg[\frac{N^2-1}{12} + \frac{n(n-N)}{2} \Bigg] h^n$$

$$- \frac{1}{2} \Bigg[\frac{N-1}{2} - n \Bigg] h^{n+1} \frac{\partial}{\partial h} + \frac{1}{4} h^{n+2} \frac{\partial^2}{\partial h^2} \Bigg\} \Bigg\{ \frac{\Psi_n(\eta_1,...,\eta_M|h)}{h^n} \Bigg\} \quad . \tag{125}$$

It remains only to rewrite this expression in terms of the $\uparrow$ coordinates $z_1, ..., z_M$. We have

$$h^{n+1} \frac{\partial}{\partial h} \Big\{ \frac{\Psi_n(\eta_1, ..., \eta_M | h)}{h^n} \Big\} = \sum_j^M \frac{h}{h - \eta_j} \Psi_n(\eta_1, ..., \eta_M | h)$$

$$= \Big\{ \frac{N-1}{2} - \sum_j^M \frac{h}{h - z_j} \Big\} \Psi_n(z_1, ..., z_M | h)$$

$$= \Big\{ (\frac{N-1}{2}) h^n - h^{n+1} \frac{\partial}{\partial h} \Big\} \Big\{ \frac{\Psi_n(z_1, ..., z_M | h)}{h^n} \Big\} \quad , \tag{126}$$

and

$$h^{n+2} \frac{\partial^2}{\partial h^2} \Big\{ \frac{\Psi_n(\eta_1, ..., \eta_M | h)}{h^n} \Big\} = \sum_{j \neq k}^M \frac{h^2}{(h - \eta_j)(h - \eta_k)} \Psi_n(\eta_1, ..., \eta_M | h)$$

$$= \sum_j^M \frac{h}{h - \eta_j} \Big\{ \frac{N-1}{2} - \sum_k^M \frac{h}{h - z_k} - \frac{h}{h - \eta_j} \Big\} \Psi_n(\eta_1, ..., \eta_M | h)$$

$$= \Big\{ \Big[\frac{N-1}{2} - \sum_k^M \frac{h}{h - z_k} \Big]^2 + \frac{(N-1)(N-5)}{12}$$

$$+ \sum_j^M (\frac{h}{h - z_j})^2 \Big\} \Psi_n(z_1, ..., z_M | h)$$

$$= \Big\{ \Big[\frac{(N-1)(N-2)}{3} + 2 \sum_j^M (\frac{h}{h - z_j})^2 \Big] h^n$$

$$- (N-1) h^{n+1} \frac{\partial}{\partial h} + h^{n+2} \frac{\partial^2}{\partial h^2} \Big\} \Big\{ \frac{\Psi_n(z_1, ..., z_M | h)}{h^n} \Big\} \quad , \tag{127}$$

which gives

$$\sum_{\alpha \neq \beta}^N [\frac{P c_{\alpha\uparrow} c_{\beta\uparrow}^\dagger P}{|z_\alpha - z_\beta|^2} \Psi_n](z_1, ..., z_M | h) = \Big\{ \frac{n(n-1)}{2} h^n + \sum_j^M \frac{h^{n+2}}{(h - z_j)^2}$$

$$- n\ h^{n+1} \frac{\partial}{\partial h} + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \bigg\} \bigg\{ \frac{\Psi_n(z_1, ..., z_M | h)}{h^n} \bigg\} \quad . \tag{128}$$

For the Ising and electrostatic energies we have

$$\sum_{\alpha \neq \beta}^{N} [\frac{S_\alpha^z S_\beta^z}{|z_\alpha - z_\beta|^2} \Psi_n](z_1, ..., z_M | h)$$

$$= \bigg\{ \sum_{j \neq k}^{M} \frac{1}{|z_j - z_k|^2} + \sum_{j}^{M} \frac{1}{|h - z_j|^2} - \frac{N(N^2 - 1)}{48} \bigg\} \Psi_n(z_1, ..., z_M | h) \tag{129}$$

and

$$\sum_{\alpha \neq \beta}^{N} \frac{1}{|z_\alpha - z_\beta|^2} \bigg[\frac{1}{2}(n_\alpha + n_\beta) - \frac{1}{4} n_\alpha n_\beta - \frac{3}{4} \bigg] \Psi_n(z_1, ..., z_M | h)$$

$$= \frac{1 - N^2}{24} \Psi_n(z_1, ..., z_M | h) \quad . \tag{130}$$

Adding these contributions together we obtain

$$\sum_{\alpha \neq \beta}^{N} \frac{1}{|z_\alpha - z_\beta|^2}\ P \bigg[S_\alpha^+ S_\beta^- + S_\alpha^z S_\beta^z + \sum_s c_{\alpha s} c_{\beta s}^\dagger + \frac{1}{2}(n_\alpha + n_\beta)$$

$$- \frac{1}{4} n_\alpha n_\beta - \frac{3}{4} \bigg] P\ \Psi_n(z_1, ..., z_M | h)$$

$$= \bigg[- \frac{N(N^2 - 1)}{48} + n(n - \frac{N+1}{2}) \bigg] \Psi_n(z_1, ..., z_M | h) \quad . \quad \Box \tag{131}$$

The propagating holon wavefunction differs from that of the spinon in being defined for *all* values of m, including those for which it is not an eigenstate of the $\mathcal{H}_{KY}$. For these other values of m we observe that

$$0 < \ | < \Psi_m | \sum_{\alpha}^{N} P c_{\alpha \downarrow}^\dagger P | \Psi_m^{holon} > |^2 \ < 1 \quad , \tag{132}$$

i.e. that a supersymmetric rotation of the spinon, which is an eigenstate of $\mathcal{H}_{KY}$, has a nonzero projection onto the corresponding holon. The physical meaning of this is that the holon can lose energy by spontaneous emission of

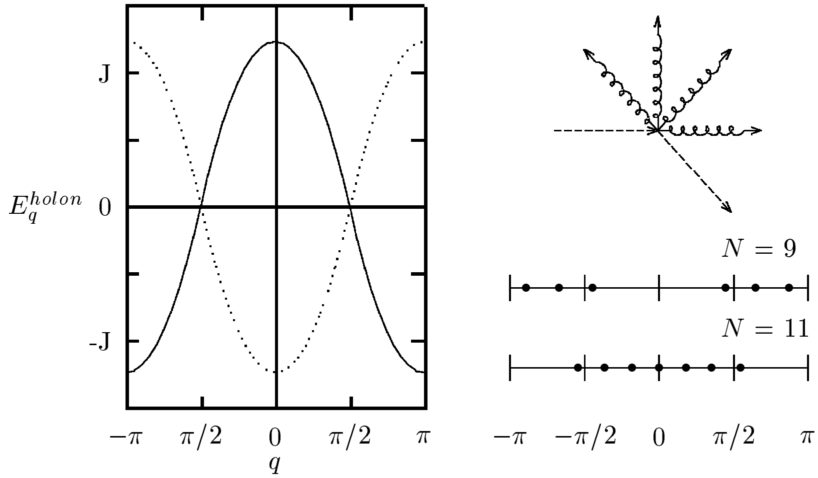

Figure 7: Left: Holon dispersion relation defined by Eq. (134). Right: Allowed values of q for adjacent odd N. Only the negative-energy holons are eigenstates of the Hamiltonian as positive-energy holons can lose energy by spontaneous emission of spinons.

spinons. Let us assign to this holon the energy of the exact eigenstate onto which it projects and write

$$E_m^{\rm holon} = \frac{J}{2}(\frac{2\pi}{N})^2 \left[\begin{array}{ll} m(\frac{N+1}{2}+m) & -\frac{N+1}{2} \leq m \leq 0 \\ m(\frac{N-1}{2}-m) & 0 \leq m \leq \frac{N-1}{2} \end{array} \right] \quad , \tag{133}$$

or

$$E_q^{holon} = \left[\begin{array}{lll} E_q & |q| < \frac{\pi}{2} & (\text{mod } 2\pi) \\ -E_q & |q| \geq \frac{\pi}{2} & (\text{mod } 2\pi) \end{array} \right] \quad , \tag{134}$$

the crystal momentum q and spinon energy E_q are defined as in Eqs. (43) and (46). This is shown in Fig. 7. The positive-energy part of the holon band is then unstable because it has negative curvature and thus does not satisfy the Landau criterion for spontaneous decay. A negative-energy holon, on the other hand, is forbidden from decaying by momentum conservation.

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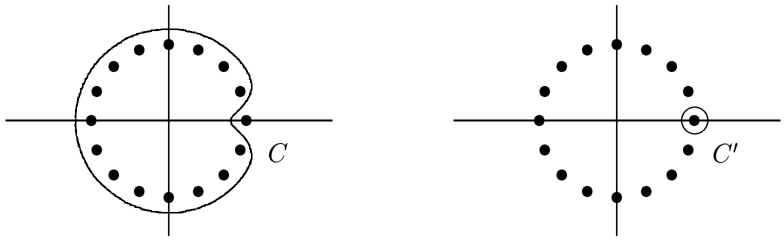

Figure 8: Contours used in Eqs. (136) and (137).

A Fourier Sums

Since the lattice sites z_α are roots of unity we have

$$\prod_{\alpha}^{N}(z - z_\alpha) = z^N - 1 \quad . \tag{135}$$

Then for $0 < m \le N$ we have

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^m}{z_\alpha - 1}$$

$$= \frac{N}{2\pi i}\oint_C \frac{z^{m-1}\,dz}{(z-1)(z^N-1)} = -\frac{N}{2\pi i}\oint_{C'} \frac{z^{m-1}\,dz}{(z-1)(z^N-1)}$$

$$= -\frac{N}{2\pi i}\oint \left\{ \frac{1 + \binom{m-1}{1} x + \binom{m-1}{2} x^2 + \ldots}{\binom{N}{1} + \binom{N}{2} x + \binom{N}{3} x^2 + \ldots} \right\} \frac{dx}{x^2}$$

$$= \frac{N+1}{2} - m \quad , \tag{136}$$

and

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^m}{|z_\alpha - 1|^2} = -\sum_{\alpha=1}^{N-1} \frac{z_\alpha^{m+1}}{(z_\alpha - 1)^2}$$

$$= -\frac{N}{2\pi i}\oint_C \frac{z^m\,dz}{(z-1)^2(z^N-1)} = \frac{N}{2\pi i}\oint_{C'} \frac{z^m\,dz}{(z-1)^2(z^N-1)}$$

$$= \frac{1}{2\pi i} \oint \left\{ \frac{1 + \binom{m-1}{1} x + \binom{m-1}{2} x^2 + \ldots}{\binom{N}{1} + \binom{N}{2} x + \binom{N}{3} x^2 + \ldots} \right\} \frac{dx}{x^3}$$

$$= \frac{N^2 - 1}{12} - \frac{m(N-1)}{2} + \frac{m(m-1)}{2} \quad . \tag{137}$$

B Problems

1. Let $\eta_1, ..., \eta_{N/2}$ be the sites complementary to $z_1, ..., z_{N/2}$. Show that

$$\prod_{j<k}^{N/2} (z_j - z_k)^2 \prod_{j}^{N/2} z_j = \prod_{j<k}^{N/2} (\eta_j - \eta_k)^2 \prod_{j}^{N/2} \eta_j \quad . \tag{138}$$

2. Show that for any polynomial $p(z)$ of degree less than N

$$\sum_{\beta \neq \alpha} \frac{p(z_\beta)}{z_\alpha - z_\beta} = \frac{1}{z_\alpha} \left[N p(0) - \left(\frac{N+1}{2} \right) p(z_\alpha) \right] + \frac{\partial p}{\partial z}(z_\alpha) \quad . \tag{139}$$

3. Show that for $\alpha \neq \beta \neq \gamma$

$$(\vec{S}_\alpha \times \vec{S}_\beta)^\dagger = (\vec{S}_\alpha \times \vec{S}_\beta) \tag{140}$$

$$\vec{S}_\alpha \cdot (\vec{S}_\beta \times \vec{S}_\gamma) = \vec{S}_\gamma \cdot (\vec{S}_\alpha \times \vec{S}_\beta) = -\vec{S}_\alpha \cdot (\vec{S}_\gamma \times \vec{S}_\beta) \tag{141}$$

$$[(\vec{S}_\alpha \cdot \vec{S}_\beta), (\vec{S}_\alpha \times \vec{S}_\beta)] = 0 \tag{142}$$

$$[(\vec{S}_\alpha \cdot \vec{S}_\gamma), (\vec{S}_\alpha \times \vec{S}_\beta)] = i \left[(\vec{S}_\alpha \cdot \vec{S}_\beta) \vec{S}_\gamma - (\vec{S}_\gamma \cdot \vec{S}_\beta) \vec{S}_\alpha \right] \quad . \tag{143}$$

4. Show that the operator

$$\vec{R}_\alpha(z)$$

$$= \sum_{\beta \neq \alpha} \left[\left(\frac{1-z}{2} \right) \frac{z_\alpha}{z_\alpha - z_\beta} + \left(\frac{1+z}{2} \right) \frac{z_\beta}{z_\alpha - z_\beta} \right] [i(\vec{S}_\alpha \times \vec{S}_\beta) + \vec{S}_\beta] \quad , \tag{144}$$

where z is an arbitrary complex number, satisfies

$$\vec{R}_\alpha(z) | \Psi >= 0 \quad , \tag{145}$$

where $|\Psi>$ is the Haldane-Shastry ground state, and

$$\sum_\alpha \vec{R}_\alpha^\dagger(z) \cdot \vec{R}_\alpha(z) = \frac{3}{2} \Bigg\{ 3 \sum_{\alpha \neq \beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{N(N^2+5)}{16}$$

$$+\left[\frac{N-1}{4}\left(|z|^2-1\right)-\frac{1}{2}\right]S^2-\left(\frac{z+z^*}{2}\right)\vec{S}\cdot\vec{\Lambda}\Bigg\} \quad . \tag{146}$$

5. Show that the ground state wavefunction for a noninteracting fermi sea on this lattice is

$$\Phi(z_1,...,z_{N/2},\eta_1,...,\eta_{N/2})=\prod_j^{N/2}(z_j\eta_j)^{-N/2}\prod_{j\leq k}^{N/2}(z_j-z_k)(\eta_j-\eta_k) \quad . \tag{147}$$

Then show that the Haldane-Shastry ground state is the Gutzwiller projection of $|\Phi>$, i.e. that $|\Psi_{HS}>=P\ |\Phi>$, where P is defined as in Eq. (107). Hint: If p denotes a permutation of $N/2$ things and sgn(p) is its sign, then

$$\sum_p^{(N/2)!}\text{sgn}(p)\ z_{p(1)}^0\times...\times z_{p(N/2)}^{N/2-1}=\prod_{j<k}^{N/2}(z_j-z_k) \quad . \tag{148}$$

6. For lattice site z_α different from 1 show that

$$\frac{2}{J}(\frac{N}{2\pi})^2\sum_{m=-(N+1)/2}^{(N-1)/2}E_m^{\text{holon}}z_\alpha^m=\frac{z_\alpha}{(1-z_\alpha)^2}\Bigg\{\frac{z_\alpha^{(N+1)/2}-z_\alpha^{(N-1)/2}}{2}$$

$$+\left[\frac{1+z_\alpha}{1-z_\alpha}+\frac{1}{2}\right]\left[z_\alpha^{(N+1)/2}+z_\alpha^{(N-1)/2}-2\right]\Bigg\} \quad , \tag{149}$$

where E_m^{holon} is given by Eq. (133). Then use this result to show that

$$<\psi_\alpha|\psi_\beta>=\sum_{z_1}...\sum_{z_M}\psi_\alpha^*(z_1,...z_M)\psi_\beta(z_1,...,z_M)$$

$$=\frac{2<\psi_\alpha|\psi_\alpha>}{N}\Bigg\{\frac{1-(z_\alpha^*z_\beta)^{(N+1)/2}}{1-z_\alpha^*z_\beta}-\frac{1+(z_\alpha^*z_\beta)^{(N-1)/2}}{4}\Bigg\} \quad , \tag{150}$$

where $|\psi_\alpha>$ and $|\psi_\beta>$ are localized spinon wavefunctions defined per Eq. (34). Hint: The fourier sum measures $<\psi_\alpha|c_{\alpha\downarrow}^\dagger\mathcal{H}_{KY}c_{\beta\downarrow}|\psi_\beta>$.

7. The amplitudes a_ℓ defined by Eq. (50) are the right eigenvectors of the non-hermitian matrix of Eq. (58). Show that the left eigenvectors of this matrix are

$$b_\ell = \frac{1}{2\ell[\ell+m-n-1/2]} \sum_{k=\ell+1}^{0} (m-n+2k) b_k \qquad (b_0 = 0) \quad , \qquad (151)$$

for $\ell \leq 0$. The size of these "adjoint" coefficients may be judged from Fig. 5, where we have plotted the function $\bar{f}_{mn}(z) = \sum_\ell b_\ell z^\ell$. Then show that

$$\sum_\alpha^N z_\alpha^k \, S_\alpha^- |\psi> = \sum_{j=0}^{k/2} c_j \; |\psi_{k-j,j}> \quad , \qquad (152)$$

where $|\psi>$ is the Haldane-Shastry ground state $|\psi_{mn}>$ is the 2-spinon eigenstate defined by Eq. (48), and c_j are a set of coefficients. Obtain an expression these in terms of the b_ℓ.

8. Let A_β and B_β be any quantum operators and let

$$a(z) = \left[\frac{z^{N/2} - z^{-N/2}}{2N}\right] \sum_\beta^N \left(\frac{z+z_\beta}{z-z_\beta}\right) A_\beta$$

$$b(z) = \left[\frac{z^{N/2} - z^{-N/2}}{2N}\right] \sum_\beta^N \left(\frac{z+z_\beta}{z-z_\beta}\right) B_\beta \quad . \qquad (153)$$

Show that

$$\frac{N}{2\pi i} \oint a^\dagger(z) b(z) \, \frac{dz}{z} = \sum_\beta^N A_\beta^\dagger B_\beta \quad . \qquad (154)$$

9. Show that the operator $\sigma(z)$ defined by Eq. (101) is hermitian whenever $|z| = 1$.

10. Let N be odd and let $z_1, ..., z_M$ and $\eta_1, ..., \eta_M$ with $M = (N-1)/2$ be distinct lattice sites. Show that

$$\sum_{j \neq k}^M \frac{1}{|\eta_j - \eta_k|^2} = \sum_{j \neq k}^M \frac{1}{|z_j - z_k|^2} + \sum_{j=1}^M \frac{1}{|h - z_j|^2} - \frac{N^2-1}{12} \quad , \qquad (155)$$

where h denotes the one site not occupied by z_j or η_j.

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